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A HAMILTON–JACOBI TYPE EQUATION IN CONTROL PROBLEMS WITH HEREDITARY INFORMATION†

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In the problem of conflict control with a each of information, formalized as a differential game [1–11] in a system with delays, a Hamilton–Jacobi type equation is written for the value functional using the concept of co-invariant derivatives [12]. Like that set out earlier [10], the corresponding generalized minimax solution of this equation is considered. Control strategies extremal to the given solution are constructed. It is proved that these strategies form a saddle point of the game, while the value functional is identical with the minimax solution of the equation. © 2000 Elsevier Science Ltd. All rights reserved.

1. A DIFFERENTIAL GAME WITH HEREDITARY INFORMATION

Consider a system with delay

$$
\begin{gathered}
dx[t]/dt = f(t, x[t_*[\cdot]t], u, v), \quad t_* \leqslant t_0 \leqslant t \leqslant T \\
x \in R^n, \quad u \in U \subset R^r, \quad v \in V \subset R^m
\end{gathered}
\tag{1.1}
$$

where x is a phase vector, u and v are control actions of the first and second players respectively, t_*, t_0 and T are specified instants of time ($t_0 < T$), U and V are known compacts, and $x[t_*[\cdot]t] = \{x[\tau], t_* \leqslant \tau \leqslant t\}$ is the history of motion that has occurred up to the instant of time t.

Let $C([t_*, T], R^n)$ be the space of continuous functions $x(\cdot) = \{x[t] \in R^n, t_* \leqslant t \leqslant T\}$.

We will denote by G the set of pairs $g = (t, x[t_*[\cdot]t])$, for which $t_0 \leqslant t \leqslant T$, and $x[t_*[\cdot]t]$ is a part (from t_* to t) of a certain function from $C([t_*, T], R^n)$. On G we shall define the metric

$$
\rho(g_1, g_2) = \max\{\rho^*(g_1, g_2), \ \rho^*(g_2, g_1)\}
\tag{1.2}
$$

where

$$
g_1 = (t_1, x^{(1)}[t_*[\cdot]t_1]) \in G, \quad g_2 = (t_2, x^{(2)}[t_*[\cdot]t_2]) \in G
$$

$$
\rho^*(g_{i+1}, g_{2-i}) = \max_{t_* \leqslant \xi \leqslant t_{i+1}} \ \min_{t_* \leqslant \eta \leqslant t_{2-i}} \max\{|\xi - \eta|, \|x^{(i+1)}[\xi] - x^{(2-i)}[\eta]\|\}, \quad i = 0, 1
$$

Here and below, $\|\cdot\|$ is the vector Euclidean norm.

Below, the properties of continuity with respect to $(t, x[t_*[\cdot]t])$ are understood with respect to changes in the metric $\rho(\cdot, \cdot)$ [Eq (1.2)].

For example, the function $z(t, x[t_*[\cdot]t]) : G \to R^n$ will be continuous in G if, for any $g^* = (t^*, x^*[t_*[\cdot]t^*]) \in G$ and $\varepsilon > 0$, $\delta > 0$ is found such that, for all $g = (t, x[t_*[\cdot]t])$ [$\in G$, such that $\rho(g, g^*) \leqslant \delta$, the following inequality is satisfied

$$
\| z(t^*, x^*[t_*[\cdot]t^*]) - z(t, x[t_*[\cdot]t]) \| \leqslant \varepsilon
$$

In (1.1) the function $f(t, x[t_*[\cdot]t], u, \upsilon) \in R^n$ is defined on $G \times U \times V$ and satisfies the following requirements.

(1_f) The function $f(t, x[t_*[\cdot]t], u, \upsilon)$ is continuous with respect to the set of their arguments on $G \times U \times V$.

(2_f) For any compact $D \subset C([t_*, T], R^n)$ a there is a number $\Lambda > 0$ such that, for all $t \in [t_0, T]$, $u \in U$, $\upsilon \in V$ and $x(\cdot) \in D, x''(\cdot) \in D$, the following estimate (the Lipschitz condition with respect to $x[t_* [\cdot]t]$) holds

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$$\| f(t,x'[t_*[\cdot]t],u,\upsilon)-f(t,x''[t_*[\cdot]t],u,\upsilon)\| \leqslant \Lambda \max_{t_*\leqslant\tau\leqslant t} \| x'[\tau]-x''[\tau] \|$$

(3_f) A number $\varkappa > 0$ exists such that, at all $(t, x[t_*[\cdot]t], u, \upsilon) \in G \times U \times V$, the following inequality is satisfied

$$\| f(t,x[t_*[\cdot]t],u,\upsilon)\| \leqslant \varkappa(1+\max_{t_*\leqslant\tau\leqslant t} \| x[\tau] \|)$$

(4_f) For any $(t, x[t_*[\cdot]t]) \in G$ and $s \in R^n$, the following equality exists (the saddle point condition in a small game [1–3]).

$$\min_{u\in U}\max_{\upsilon\in V}\langle s, f(t,x[t_*[\cdot]t],u,\upsilon)\rangle = \max_{\upsilon\in V}\min_{u\in U}\langle s, f(t,x[t_*[\cdot]t],u,\upsilon)\rangle$$

Here and below, $\langle\cdot,\cdot\rangle$ is the scalar product of vectors.

Suppose the initial state

$$g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G, \quad t^0 < T$$

of system (1.1) is specified. The Borel measurable realizations

$$u[t^0[\cdot]T) = \{u[t] \in U,\ t^0 \leqslant t < T\},\ \upsilon[t^0[\cdot]T) = \{\upsilon[t] \in V, t^0 \leqslant t < T\}$$

are admissible.

With conditions (1_f)–(3_f), from state g^0 such realizations naturally give rise to the motion of system (1.1)—the function $x(\cdot) \in C([t_*, T], R^n)$, which is identical with $x^0[t_*[\cdot]t^0]$ on $[t_*, t^0]$ and, when $t \in [t^0, T]$, satisfies the equality

$$x[t] = x[t^0] + \int_{t^0}^{t} f(\tau, x[t_*[\cdot]\tau],\ u[\tau],\ \upsilon[\tau])d\tau$$

where the integral is understood in the Lebesgue sense.

The quality index γ of this motion is specified by the continuous functional $\sigma : C([t_0, T], R^n) \mapsto R$, so that

$$\gamma = \sigma(x[t_0[\cdot]T]) \tag{1.3}$$

The objective of the first player is to minimize γ, and that of the second is to maximize it.

The control strategies of the first and second players will be termed arbitrary functions $u(t, x[t_*[\cdot]t]) \in U$ and $\upsilon(t, x[t_*[\cdot]t]) \in V$, respectively, where $(t, x[t_*[\cdot]t]) \in G$ and $t \in [t^0, T]$. The selected strategies $u(\cdot)$ and subdivision

$$\Delta_\delta = \{t_i : t_1 = t^0,\ 0 < t_{i+1} - t_i \leqslant \delta, i = 1,\dots,k, t_{k+1} = T\} \tag{1.4}$$

of the time segment $[t^0, T]$ form a step-by-step control law for the first player $(u(\cdot), \Delta_\delta)$ which, forming the realization $u[t^0[\cdot]T)$ according to the law

$$u[t] = u(t_i, x[t_*[\cdot]t_i]),\quad t_i \leqslant t < t_{i+1},\quad i = 1,\dots,k \tag{1.5}$$

in a pair with the admissible realization $\upsilon[t^0[\cdot]T)$, generates, from the initial state $(t^0, x^0[t_*[\cdot]t^0])$, the unique motion $x(.\ | t^0, x^0[t_*[\cdot]t^0], u(\cdot), \Delta_\delta, \upsilon[t^0[\cdot]T))$.

We shall define the quantity

$$\Gamma_u(t^0, x^0[t_*[\cdot]t^0],\ u(\cdot)) = \limsup_{\delta\downarrow 0}\ \sup_{\Delta_\delta}\ \sup_{\upsilon[t^0[\cdot]T)} \sigma(x([t_0[\cdot]T \mid t^0,\ x^0[t_*[\cdot]t^0],\ u(\cdot),\ \Delta_\delta,\ \upsilon[t^0[\cdot]T))) \tag{1.6}$$

which is termed the guaranteed result of the strategy $u(\cdot)$ of the first player. The optimum guaranteed result of the first player will be

$$\Gamma_u^0 = \Gamma_u^0(t^0, x^0[t_*[\cdot]t^0]) = \inf_{u(\cdot)} \Gamma_u(t^0, x^0[t_*[\cdot]t^0], u(\cdot)) \tag{1.7}$$

The values of the guaranteed result of the strategy $\upsilon(\cdot)$ of the second player and the optimum guaranteed result of the second player are determined in a similar way (the replacement in (1.6) and (1.7) of the symbols u, υ, sup and inf by υ, u, inf and sup, respectively). From this it follows that, for any initial state $(t^0, x^0[t_*[\cdot]t^0]) \in G$, the following inequality holds

$$\Gamma_u^0 \geqslant \Gamma_\upsilon^0 \tag{1.8}$$

When equality occurs in (1.8), the differential game with hereditary information (1.1), (1.3) is said to have the value

$$\Gamma^0(t^0, x^0[t_*[\cdot]t^0]) = \Gamma_u^0 = \Gamma_\upsilon^0$$

in the class of pure strategies. Here, if inf and sup are achieved in (1.7) and in the analogous expression for Γ_υ^0, i.e.

$$\Gamma_u^0 = \Gamma_u(t^0, x^0[t_*[\cdot]t^0],\ u^0(\cdot)),\ \Gamma_\upsilon^0 = \Gamma_\upsilon(t^0, x^0[t_*[\cdot]t^0],\ \upsilon^0(\cdot))$$

then the pair of strategies $(u^0(\cdot), \upsilon^0(\cdot))$ is said to form a saddle point of game (1.1), (1.3) and these strategies are termed optimal.

Further the functional Hamilton–Jacobi type equation in co-invariant derivatives [12] corresponds to the differential game with hereditary information (1.1), (1.3) considered. Along with difficulties related to the analysis of normal Hamilton–Jacobi equations [10, 11, 13, 14], this equation is complicated by features resulting from the delay effect. Like the approach described earlier [10], the corresponding definition of the generalized minimax solution of this equation is given. Under conditions (1_f)–(4_f), the given minimax solution exists, it is unique and correct. A method is indicated for constructing the optimum strategies, and it is proved that the functional $\Gamma^0 : G \mapsto R$ of the value of game (1.1), (1.3) is identical with the minimax solution of the associated equation.

2. THE FUNCTIONAL HAMILTON–JACOBI TYPE EQUATION IN CO-INVARIANT DERIVATIVES

Consider the functional

$$\varphi(g) = \varphi(t, x[t_*[\cdot]t]) : G \mapsto R$$

Let $g^* = (t^*, x^*[t_*[\cdot]t^*]) \in G$ $(t^* < T)$, Lip (g^*) be a set of functions $y(\cdot) \in C([t_*, T], R^n)$ identical with $x^*[t_*[\cdot]t^*]$ on $[t_*, t^*]$, each of which, with a certain (its own) constant, satisfies the Lipschitz condition in $[t^*, T]$. We shall say [12] that the functional φ is co-invariantly differentiable at the point g^* with respect to Lip(g^*) (ci-differentiable in g^*) if a number $\partial_t\varphi(g^*)$ and an n-vector $\nabla\varphi(g^*)$ exist such that, for any function $y(\cdot) \in$ Lip(g^*), the following equality is satisfied

$$\begin{aligned}&\varphi(t^* + \xi, y[t_*[\cdot]t^* + \xi]) - \varphi(t^*, x^*[t_*[\cdot]t^*]) = \partial_t\varphi(g^*)\xi + \langle\nabla\varphi(g^*), y[t^* + \xi] - x^*[t^*]\rangle + \\ &+ o_{y(\cdot)}(\xi), \quad \xi \in [0,\ T - t^*]\end{aligned} \tag{2.1}$$

where $o_{y(\cdot)}(\xi)$ depends on the choice of $y(\cdot) \in$ Lip(g^*) $(o_{y(\cdot)}(\xi)/\xi \to 0$ when $\xi \to +0)$.

The quantities $\partial_t\varphi(g^*)$ and $\nabla\varphi(g^*)$ will be termed co-invariant derivatives with respect to t and the gradient, respectively, of the functional φ at the point g^*. The functional φ will be termed ci-differentiable on G if it is ci-differentiable at each point $g = (t, x[t_*[\cdot]t] \in G(t < T)$. If in this case the quantities $\partial_t\varphi(g)$ and $\nabla\varphi(g)$ are continuous, we shall say that the functional φ is continuously ci-differentiable.

More detailed information on the properties, the computational methods and the applications of the co-invariant derivatives of the functionals can be found, for example, in [12].

Remark. The definition of the co-invariant derivatives is given above [see (2.1)] for functionals $\varphi(t, x[t_*[.]t])$, defined, generally speaking, only on continuous functions $x[t_*[\cdot]t])$, and it therefore differs from the corresponding definition given earlier [12, pp. 28–50], where co-invariant derivatives are naturally introduced for functionals defined on piecewise-continuous functions. Nevertheless, if some functional $\varphi(t, x[t_*[\cdot]t])$, defined on piecewise-continuous functions, is ci-differentiable at the point $g^* = (t^*, x^*[t_*[\cdot]t]) \in G$ with respect to Lip(g^*) in the sense of [12], then its contraction φ_G on continuous functions will be ci-differentiable at g^* in the sense of (2.1), and here the corresponding co-invariant derivatives will be identical.

For system (1.1) we shall define the Hamiltonian $H(t, x[t_*[\cdot]t], s)$: $G \times R^n \mapsto R$ according to the equality

$$H(t,x[t_*[\cdot]t],s)=\max_{v\in V}\min_{u\in U}\langle s,f(t,x[t_*[\cdot]t],\ u,\ v\ )\rangle \tag{2.2}$$

Let us consider the Hamilton–Jacobi type equation in co-invariant derivatives

$$\partial_t\varphi(t,x[t_*[\cdot]t])+H(t,x[t_*[\cdot]t],\ \ \nabla\varphi(t,x[t_*[\cdot]t]))=0,\ \ (t,\ x[t_*[\cdot]t])\in G,\ \ t<T \tag{2.3}$$

with the condition at the right-hand end

$$\varphi(T,x[t_*[\cdot]T])=\sigma(x[t_0[\cdot]T]),\ \ \ x(\cdot)\in C([t_*,\ T],\ R^n) \tag{2.4}$$

Note that, with requirements (1_f)–(4_f), Hamiltonian (2.2) will satisfy the following conditions.

(1_H) For any $s\in R^n$, functional $(t,x[t_*[\cdot]t],\mapsto H(t,x[t_*[\cdot]t],s)$ is continuous on G.

(2_H) For any compactum $D\subset C([t^*,T],R^n)$ a number $\Lambda>0$ exists such that, for all $t\in[t_0,T]$, $s\in R^n$, $\|s\|=1$ and $x'(\cdot)\in D, x''(\cdot)\in D$, the following estimate (the Lipschitz condition with respect to $x[t_*[\cdot]t]$) holds

$$|H(t,x'[t_*[\cdot]t],\ s)-H(t,x''[t_*[\cdot]t],\ s)|\leqslant\Lambda\max_{t_*\leqslant\tau\leqslant t}\|x'[\tau]-x''[\tau]\|$$

(3_H) For any $(t,x[t_*[\cdot]t])\in G$ and $s',s''\in\{s\in R^n:\ \|s\|\leqslant 1\}$, the following inequality (the Lipschitz condition with respect to s) holds

$$|H(t,x[t_*[\cdot]t],s')-H(t,x[t_*[\cdot]t],s'')|\leqslant L(t,x[t_*[\cdot]t])\|s'-s''\|$$

where $L(t,x[t_*[\cdot]t])$ is a functional that is continuous on G and satisfies the estimate

$$L(t,x[t_*[\cdot]t])\leqslant\varkappa(1+\max_{t_*\leqslant\tau\leqslant t}\|x[\tau]\|),\ \ (t,x[t_*[\cdot]t])\in G,\ \ \varkappa=\text{const}>0$$

(4_H) For any $(t,x[t_*[\cdot]t])\in G$, the function $s\mapsto H(t,x[t_*[\cdot]t],s)$ is positively homogeneous, i.e.

$$H(t,x[t_*[\cdot]t],\alpha s)=\alpha H(t,x[t_*[\cdot]t],s),\ \ \alpha\geqslant 0$$

Under these conditions, a ci-differentiable functional satisfying relations (2.3) and (2.4) cannot exist. As in the case of the normal Hamilton–Jacobi equations, in problem (2.3), (2.4) the need arises to determine a suitable generalized equation.

We shall define the minimax solution of problem (2.3), (2.4).

Suppose P and Q are certain non-empty sets (to fix our ideas it can be assumed that P and Q are subsets of certain finite-dimensional spaces), and the multivalued mappings

$$(t,x[t_*[\cdot]t],q)\mapsto F^*(t,x[t_*[\cdot]t],\ q)\subset R^n$$

$$(t,x[t_*[\cdot]t],p)\mapsto F_*(t,x[t_*[\cdot]t],\ p)\subset R^n$$

where $(t,x[t_*[\cdot]t],p,q)\in G\times P\times Q$, satisfy the following requirements.

(1_K) For any $(t,\ x[t_*[\cdot]t])\in G, p\in P$ and $q\in Q$, the sets $F^*(t,\ x[t_*[\cdot]t],\ q)$ and $F_*(t,\ x[t_*[\cdot]t],\ p)$ are non-empty convex compactum in R^n. A number $a>0$ also exists such that the following estimate holds.

$$\max\{\|f\|\,|\,f\in F^*(t,x[t_*[\cdot]t],q)\cup F_*(t,x[t_*[\cdot]t],p)\}\leqslant a(1+\max_{t_*\leqslant\tau\leqslant t}\|x[\tau]\|)$$

$$(t,x[t_*[\cdot]t],p,q)\in G\times P\times Q$$

(2_K) For any $p\in P$ and $q\in Q$, the multivalued mappings

$$(t,x[t_*[\cdot]t])\mapsto F^*(t,x[t_*[\cdot]t],q),\ \ \ (t,x[t_*[\cdot]t])\mapsto F_*(t,x[t_*[\cdot]t],p)$$

are semicontinuous from above the inclusion.

(3_K) For any $(t,x[t_*[\cdot]t])\in G$ and $s\in R^n$, the following equalities hold

$$\sup_{q\in Q}\ \min_{f\in F^*(t,x[t_*[\cdot]t],q)}\langle s,f\rangle=H(t,x[t_*[\cdot]t],s)=\inf_{p\in P}\ \max_{f\in F_*(t,x[t_*[\cdot]t],p)}\langle s,f\rangle$$

The set of pairs $\{Q, F^*(\cdot)\}$ ($\{P, F_*(\cdot)\}$) satisfying requirements (1_K)–(3_K) will be denoted by $K^*(H)(K_*(H))$.

Remark. With conditions (1_H)–(4_H), $K^*(H) \neq \theta, K^*(H) \neq \theta$. In particular, requirements (1_K)–(3_K) are satisfied

$$P = Q = R^n$$

$$F^*(t, x[t_*[\cdot]t], q) = \{f \in F(t, x[t_*[\cdot]t]) : \langle f, q\rangle \geq H(t, x[t_*[\cdot]t], q)\}$$

$$F_*(t, x[t_*[\cdot]t], p) = \{f \in F(t, x[t_*[\cdot]t]) : \langle f, p\rangle \leq H(t, x[t_*[\cdot]t], p)\}$$

where $F(t, x[t_*[\cdot]t]) = \{f \in R^n : \|f\| \leq \sqrt{2}L(t, x[t_*[\cdot]t])\}$, $L(t, x[t_*[\cdot]t])$ from condition (3_H).

Suppose

$$\{Q, F^*(\cdot)\} \in K^*(H), \quad \{P, F_*(\cdot)\} \in K_*(H)$$

We shall consider differential inclusions with delay

$$dx[t]/dt \in F^*(t, x[t_*[\cdot]t], q) \tag{2.5}$$

$$dx[t]/dt \in F_*(t, x[t_*[\cdot]t], p) \tag{2.6}$$

Let

$$g^0 = (t^0, \ x^0[t_*[\cdot]t^0]) \in G, \quad p \in P, \quad q \in Q$$

The solution of inclusion (2.5) [and accordingly (2.6)] with the initial condition g^0 will be understood, with a fixed value of $q(p)$, to be the function $x(\cdot) \in C([t_*, T], R^n)$, identical with $x^0(t_*[\cdot]t^0]$ for $[t_*, t^0]$ on an absolutely continuous function for $[t^0, T]$ and, for almost all $t \in [t^0, T]$, satisfying inclusion (2.5) [and accordingly (2.6)]. The set of all such solutions will be denoted by $X^*(t^0, x^0[t_*[\cdot]t^0], q\,|F^*)(X_*(t^0, x^0[t_*[\cdot]t^0]p\,|F_*))$. By virtue of conditions (1_K) and (2_K), these sets will be non-empty compactum in $C([t_*, T], R^n)$ for any $(t^0, x^0 > [t_*[\cdot]t^0], p, q) \in G \times P \times Q$.

Definition. The minimax solution (MS) of problem (2.3), (2.4) will be the continuous functional $\varphi : G \mapsto R$, which satisfies boundary condition (2.4) and, for certain $\{Q, F^*(\cdot)\} \in K^*(H)$, $\{P, F_*(\cdot)\} \in K_*(H)$, the pair of inequalities

$$\sup_{(t^0, x^0[t_*[\cdot]t^0]; t, q)} \min_{x^*(\cdot)} [\varphi(t, x^*[t_*[\cdot]t]) - \varphi(t^0, x^0[t_*[\cdot]t^0])] \leq 0 \tag{2.7}$$

where

$$(t^0, x^0[t_*[\cdot]t^0]) \in G, \quad t \in [t^0, T], \quad q \in Q, \quad x^*(\cdot) \in X^*(t^0, x^0[t_*[\cdot]t^0], \ q\,|\,F^*)$$

and

$$\inf_{(t^0, x^0[t_*[\cdot]t^0]; t, p)} \max_{x_*(\cdot)} [\varphi(t, x_*[t_*[\cdot]t]) - \varphi(t^0, x^0[t_*[\cdot]t^0])] \geq 0 \tag{2.8}$$

where

$$(t^0, x^0[t_*[\cdot]t^0]) \in G, \quad t \in [t^0, T], \quad p \in P, \quad x_*(\cdot) \in X_*(t^0, x^0[t_*[\cdot]t^0], \ p\,|\,F_*)$$

It can be shown that, under conditions (1_H)–(4_H) imposed on the Hamiltonian H, and with the continuous boundary function s. An MS of problem (2.3), (2.4) exists, it is unique and possesses the following properties. It satisfies inequalities (2.7) and (2.8) for any $\{Q, F^*(\cdot)\} \in K^*(H)$, $\{P, F_*(\cdot)\} \in K_*(H)$. It is correct, i.e. continuously depends on the variation in the boundary condition and the Hamiltonian. The MS of problem (2.3) and (2.4) may not be ci-differentiable on G, but, on the one hand, at each point $(t, x[t_*[\cdot]t])$, where this solution is ci-differentiable, it satisfies Eq. (2.3), while on the other hand the continuous functional, continuously ci-differentiable on G and satisfying relations (2.3) and (2.4), is an MS.

Remark. These assertions are substantiated chiefly by a plan set out earlier [10, pp. 13–40], taking into account features associated with the delay effect and with the functionality of the argument of the solution required. The corresponding discussion beyond the scope of the present paper and is not give here.

3. EXTREMAL STRATEGIES

We define the multivalued mappings

$$F^u(t, x[t_*[\cdot]t], \upsilon) = \mathrm{co}\{f = f(t, x[t_*[\cdot]t], u, \upsilon) \mid u \in U\}$$
$$F_\upsilon(t, x[t_*[\cdot]t], u) = \mathrm{co}\{f = f(t, x[t_*[\cdot]t], u, \upsilon) \mid \upsilon \in V\} \tag{3.1}$$
$$(t, x[t_*[\cdot]t], u, \upsilon) \in G \times U \times V$$

where F is the convex shell of set F in R^n.

Taking of conditions $(1_f$–$4_f)$ and equality (2.2) into account, it can be shown that for

$$P = U,\quad Q = V,\quad F^*(\cdot) = F^u(\cdot),\quad F_*(\cdot) = F_\upsilon(\cdot)$$

requirements (1_K)–(3_K) are satisfied [here, (1_K) with $a = \varkappa$ from (3_f)], so that

$$\{V, F^u(\cdot)\} \in K^*(H),\quad \{U, F_\upsilon(\cdot)\} \in K_*(H)$$

Note, further, that multivalued mappings

$$(t, x[t_*[\cdot]t]) \mapsto F^u(t, x[t_*[\cdot]t], \upsilon),\quad (t, x[t_*[\cdot]t]) \mapsto F_\upsilon(t, x[t_*[\cdot]t], u)$$

are equicontinuous (on the inclusion) on G with respect to $\upsilon \in V$ and $u \in U$.

Let $\varphi(t, x[t_*[\cdot]t])$ be the minimax solution of problem (2.2)–(2.4).

Then, according to Section 2, the functional φ, in particular, satisfies inequalities (2.7) and (2.8), respectively, with $Q = V$, $F^*(\cdot) = F^u(\cdot)$ and $P = U, F_*(\cdot) = F_\upsilon(\cdot)$.

Remark. Under these conditions, inequalities (2.7) and (2.8) express, respectively, the due properties of u and υ stability [3–8] of the value functional of differential game (1.1), (1.3).

Let $g^0 = (t^0, x^0[t_*[\cdot]t^0]) \in G(t^0 < T)$ be the initial state of system (1.1). We shall denote by $X^0 = X(t^0, x^0[t_*[\cdot]t^0])$ the set of solutions stemming from g^0 of the differential inclusion

$$dx[t]/dt \in \{f \in R^n : \| f \| \le \varkappa(1 + \max_{t_* \le \tau \le t} \| x[\tau] \|)\} \tag{3.2}$$

where $\varkappa$ is from condition (3_f). The set X^0 will be a non-empty compactum in $C([t_*, T], R^n)$. We shall assume that

$$W_u(t) = \{w(\cdot) \in X^0 : \varphi(t, w[t_*[\cdot]t]) \le \varphi(t^0, x^0[t_*[\cdot]t^0])\}$$
$$W_\upsilon(t) = \{w(\cdot) \in X^0 : \varphi(t, w[t_*[\cdot]t]) \ge \varphi(t^0, x^0[t_*[\cdot]t^0])\} \tag{3.3}$$
$$t^0 \le t \le T$$

By virtue of inequalities (2.7) $(Q = V, F^*(\cdot) = F^u(\cdot))$, (2.8) $(P = U, F_*(\cdot) = F_\upsilon(\cdot))$, condition (1_K) $(a = \varkappa)$ and inclusion (3.2), sets $W_u(t)$ and $W_\upsilon(t)$ are non-empty for any $t \in [t^0, T]$. Functional φ is continuous on G, and therefore $W_u(t)$ and $W_\upsilon(t)$ are compact in $C([t_*, T], R^n)$.

The extremal strategy $u_e(\cdot)$ of the first player will be determined on the basis of the arbitrary sample

$$u_e(t, x[t_*[\cdot]t]) \in \arg\min_{u \in U}\{\max_{f \in F_\upsilon(t, x[t_*[\cdot]t], u)} \langle s_u^0, f\rangle\} \tag{3.4}$$

where

$$s_u^0 = x[t] - w_u^0[t],\quad w_u^0(\cdot) \in \arg\min_{w(\cdot) \in W_u(t)}\{\max_{t_* \le \tau \le t} \| x[\tau] - w[\tau] \|\} \tag{3.5}$$

The extremal strategy $\upsilon_e(\cdot)$ of the second player will be determined on the basis of the arbitrary sample

$$\upsilon_e(t, x[t_*[\cdot]t]) \in \arg\max_{\upsilon \in V}\{\min_{f \in F^u(t, x[t_*[\cdot]t], \upsilon)} \langle s_\upsilon^0, f\rangle\} \tag{3.6}$$

where

$$s_v^0 = w_v^0[t] - x[t], \quad w_v^0(\cdot) \in \arg \min_{w(\cdot)\in W_v(t)} \{ \max_{t_* \leq \tau \leq t} \| x[\tau] - w[\tau] \| \} \tag{3.7}$$

Here, $F^u(\cdot)$ and $F_\upsilon(\cdot)$ are from (3.1), and $W_u(t)$ and $W_\upsilon(t)$ are from (3.3). Requirement (1_f) is satisfied. Therefore, the necessary values of u_e and v_e in (3.4) and (3.6) exist for any $t \in [t^0, T]$.

4. THE MINIMAX SOLUTION AND THE VALUE OF THE GAME

Theorem 1. For system (1.1), let requirements (1_f)–(4_f) be satisfied, and, in (1.3), let the functional: $C([t_0, T], R^n) \mapsto R$ be continuous. Then, for any initial state

$$(t^0, \; x^0[t_*[\cdot]t^0]) \in G, \quad t^0 < T \tag{4.1}$$

the differential game with hereditary information (1.1), (1.3) has the value $\Gamma^0(t^0, x^0[t_*[\cdot]t^0])$ in the class of pure strategies. The functional $(t^0, x^0[t_*[\cdot]t^0]) \mapsto \Gamma^0(t^0, x^0[t_*[\cdot]t^0])$ is identical with the minimax solution φ of problem (2.2)–(2.4). The extremal strategies $u_e(\cdot)$ and $\upsilon_e(.)$ form the saddle point of the game.

Proof. To prove the theorem it is sufficient to show that, for any pair $(t^0, x^0[t_*[\cdot]t^0])$ satisfying condition (4.1), the following inequalities hold

$$\begin{aligned} &\Gamma_u(t^0, x^0[t_*[\cdot]t^0], u_e(\cdot)) \leq \varphi(t^0, x^0[t_*[\cdot]t^0]) \\ &\Gamma_v(t^0, x^0[t_*[\cdot]t^0], v_e(\cdot)) \geq \varphi(t^0, x^0[t_*[\cdot]t^0]) \end{aligned} \tag{4.2}$$

In fact, from these inequalities, if expression (1.7), the analogous expression for Γ_υ^0 and also inequality (1.8) are taken into account, we have the following chain of inequalities

$$\begin{aligned} &\varphi(t^0, x^0[t_*[\cdot]t^0]) \geq \Gamma_u(t^0, x^0[t_*[\cdot]t^0], \;\; u_e(\cdot)) \geq \Gamma_u^0 \geq \\ &\geq \Gamma_v^0 \geq \Gamma_v(t^0, x^0[t_*[\cdot]t^0], \; v_e(\cdot)) \geq \varphi(t^0, x^0[t_*[\cdot]t^0]) \end{aligned}$$

which proves the theorem.

The proofs of inequalities (4.2) do not differ essentially.

The scheme of the proof of the first inequality of (4.2) is given below.

Let condition (4.1) be satisfied and let $X^0 = X(t^0, x^0[t_*[\cdot]t^0])$ [see (3.2)]. We put

$$\nu(t, x[t_*[\cdot]t], \;\; w[t_*[\cdot]t]) = \exp\{-2\Lambda(t - t_0)\} \max_{t_* \leq \tau \leq t} \| x[\tau] - w[\tau] \|^2 \tag{4.3}$$

where $\Lambda > 0$ from condition (2_f) with $D = X^0$.

Lemma 1. For any pair $(t^0, x^0[t_*[\cdot]t^0])$ satisfying condition (4.1) and any $\varepsilon > 0$, $\delta^0 > 0$ exists such that the following assertion holds.

Suppose $x^*(\cdot) \in X^0$, $w^*(\cdot) \in X^0$, $t^* \in [t^0, T)$, $t \in (t^*, T]$ and $t - t^* \leq \delta^0$. Suppose $s^* = x^*[t^*] - w^*[t^*]$ and $u_e = u_e(t^*, x^*[t_*[\cdot]t^*])$ from (3.4) with $s_u^0 = s^*$, and $\upsilon_e = \upsilon_e(t^*, w^*[t_*[\cdot]t^*])$ form (3.6) with $s_\upsilon^0 = s^*$. Then, for any $x(\cdot) \in X_*(t^*, x^*[t^*[\cdot]t^*, u_e|F_\upsilon) \in X^*(t^*, w^*[t_*[\cdot]t^*], \upsilon_e|F_u)(F_u(\cdot)$ and $F_\upsilon(\cdot)$ from (3.1)), we have $x(\cdot) \in X^0$ and $w(\cdot) \in X^0$, and the following inequality holds

$$\nu(t, x[t_*[\cdot]t], \;\; w[t_*[\cdot]t]) \leq \nu(t^*, x^*[t_*[\cdot]t^*], \;\; w^*[t_*[\cdot]t^*]) + \varepsilon(t - t^*) \tag{4.4}$$

This lemma is provision using the plan set out earlier ([3, pp. 59–61]; see also [10, pp. 97–99]) and is omitted here.

We take $\eta > 0$. The functional $\sigma : C([t_0, T], R^n \mapsto R$ is continuous, and $X^0 = X(t^0, x^0[t_*[\cdot]t^0])$ is a compactum in $C([t_*, T], R^n)$. Therefore, an $\zeta > 0$ exists such that, for all $x(\cdot) \in X^0$ and $w(\cdot) \in X^0$ satisfying the condition

$$\max_{t_* \leq \tau \leq T} \| x[\tau] - w[\tau] \| \leq \zeta$$

the following inequality holds

$$|\sigma(x[t_0[\cdot]T]) - \sigma(w[t_0[\cdot]T])| \leq \eta \tag{4.5}$$

We shall assume that

$$\varepsilon = \zeta^2 \exp\{-2\Lambda(T - t_0)\}/(T - t_0) \tag{4.6}$$

For this $\varepsilon > 0$, we obtain $\delta^0 > 0$, for which the assertions of Lemma 1 are satisfied. For $\delta \in (0, \delta^0]$, we shall adopt the subdivision $\Delta_\delta(1.4)$. Consider the control law for the first player $(u_e(\cdot), \Delta_\delta)$. Suppose that some permissible realization of $\upsilon[t^0[\cdot]T)$ has occurred, and that the motion $x^*(\cdot) = x(\cdot \mid t^0, x^0[t_*[\cdot]t^0]$, $u_e(\cdot)$, Δ_δ, $\upsilon[t^0[\cdot]T))$ of system (1.1) has been realized. By virtue of requirement (3_f) and (3.2), $x^*(\cdot) \in X^0$. We put

$$\begin{gathered} \operatorname{dist}[x[t_*[\cdot]t],\ W_u(t)] = \min_{w(\cdot) \in W_u(t)} \max_{t_* \leq \tau \leq t} \| x[\tau] - w[\tau] \| \\ x(\cdot) \in X^0, \quad t \in [t^0, T] \end{gathered} \tag{4.7}$$

where $W_u(t)$ is from (3.3).

For any $i = 1, \dots, k + 1$, the following inequality is satisfied

$$\operatorname{dist}[x^*[t_*[\cdot]t_i],\ W_u(t_i)] \leq \sqrt{\varepsilon(t_i - t_0)} \exp\{\Lambda(t_i - t_0)\} \tag{4.8}$$

which can be proved by induction from $i = 1$ to $i = k + 1$.

For $i = 1$ we have $t_1 = t^0$ and $x^*[t_*[\cdot]t^0] = x^0[t_*[\cdot]t^0]$, and therefore $x^*(\cdot) \in W_u(t^0)$, and inequality (4.8) ($i = 1$) is satisfied

In addition, we shall assume that inequality (4.8) holds when $i = j$ ($1 \leq j \leq k$) and show that (4.8) is then satisfied for $i = j + 1$.

We shall assume that $w^*(\cdot) = w_u^0(\cdot)$ from (3.5) with $t = t_j$; $s^* = x^*[t_j] - w^*[t_j]$, $u_e = u_e(t_j, x^*[t_*[\cdot]t_j])$ from (3.4) with $s_u^0 = s^*$ and $v_e = \upsilon_e(t_j, w^*[t_*[\cdot]t_j])$ from (3.6) with $s_\upsilon^0 = s^*$. Then $w^*(\cdot) \in W_u(t_j) \subset X^0$ and, on the assumption of induction, and taking relation (4.3) into account, we have

$$\nu(t_j, x^*[t_*[\cdot]t_j],\quad w^*[t_*[\cdot]t_j]) \leq \varepsilon(t_j - t_0) \tag{4.9}$$

By (2.7) ($Q = V, F^*(\cdot) = F^u(\cdot)$), a function $w(\cdot) \in X^*(t_j, w^*[t_*[\cdot]t_j], \upsilon_e \mid F^u)$ exists such that

$$\varphi(t_j, w^*[t_*[\cdot]t_j]) \geq \varphi(t_{j+1}, w[t_*[\cdot]t_{j+1}])$$

Since $w^*(\cdot) \in W_u(t_j)$, then by virtue of (3.3) we have

$$\varphi(t_j, w^*[t_*[\cdot]t_j]) \leq \varphi(t^0, x^0[t_*[\cdot]t^0])$$

From this we obtain the inequality

$$\varphi(t_{j+1}, w[t_*[\cdot]t_{j+1}]) \leq \varphi(t^0, x^0[t_*[\cdot]t^0]) \tag{4.10}$$

Let

$$x(\cdot) \in X_*(t_{j+1}, x^*[t_*[\cdot]t_{j+1}],\ u_e \mid F_\upsilon)$$

By the definition of motion

$$x^*(\cdot) = x(\cdot \mid t^0,\ x^0[t_*[\cdot]t^0],\ u_e(\cdot),\ \Delta_\delta,\ \upsilon[t^0[\cdot]T))$$

[see (1.4), (1.5) and (3.4), (3.5)] and (3.1), the function $x^*(\cdot)$ for almost all $\tau \in [t_j, t_{j+1}]$ satisfies the inclusion

$$dx^*[\tau]/d\tau \in F_\upsilon(\tau, x^*[t_*[\cdot]\tau], u_e)$$

Therefore

$$x(\cdot) \in X_*(t_j, x^*[t_*[\cdot]t_j], u_e \mid F_\upsilon)$$

Now, note that, for $t^* = t_j$, $t = t_{j+1}$ and the $x^*(\cdot)$, $w^*(\cdot)$, $x(\cdot)$ and $w(\cdot)$ selected above, all the requirements of Lemma 1 are satisfied.

From (4.4) ($t^* = t_j$, $t = t_{j+1}$), taking account of relations (4.3) and (4.9) and the equality

$$x[t_*[\cdot]t_{j+1}] = x^*[t_*[\cdot]t_{j+1}]$$

we derive

$$\max_{t_* \le \tau \le t_{j+1}} \|x^*[\tau] - w[\tau]\| \le \sqrt{\varepsilon(t_{j+1} - t_0)} \exp\{\Lambda(t_{j+1} - t_0)\} \tag{4.11}$$

Since, by virtue of (3.3) and (4.10), $w(\cdot) \in W_u(t_{j+1})$, then from (4.7) ($t = t_{j+1}$) it follows that the necessary inequality (4.8) is correct with $i = j + 1$. By mathematical induction, inequality (4.8) is satisfied for any $i = 1, \ldots, k + 1$.

From (4.8) (with $i = k + 1$), if we take (4.6) into account and select the number $\zeta > 0$ [see (4.5)], it follows that a function $w_0(\cdot) \in W_u(T)$ exists for which the inequality

$$|\sigma(x^*[t_0[\cdot]T]) - \sigma(w_0[t_0[\cdot]T])| \le \eta$$

is satisfied. By relations (2.4) and (3.3) we have

$$\sigma(w_0[t_0[\cdot]T]) = \varphi(T, w_0[t_*[\cdot]T]) \le \varphi(t^0, x^0[t_*[\cdot]t^0])$$

Therefore, the following inequality holds

$$\sigma(x^*[t_0[\cdot]T]) \le \varphi(t^0, x^0[t_*[\cdot]t^0]) + \eta \tag{4.12}$$

Remember that

$$x^*(\cdot) = x(\cdot \mid t^0, x^0[t_*[\cdot]t^0],\ u_e(\cdot),\ \Delta_\delta,\ v[t^0[\cdot]T])$$

and here $\delta \in (0, \delta^0]$, Δ_δ (1.4) and the permissible realization $v[t^0[\cdot]T)$ were chosen arbitrarily. Consequently, from (4.12), by equality (1.6) it follows that

$$\Gamma_u(t^0, x^0[t_*[\cdot]t^0],\ u_e(\cdot)) \le \varphi(t^0, x^0[t_*[\cdot]t^0]) + \eta$$

Since the number $\eta > 0$ was also chosen arbitrarily, the first inequality of (4.2) to be proved follows from this.

The second inequality of (4.2) is proved in a similar way with obvious changes.

Remark. In the practical construction of extremal strategies, depending on the specific properties of system (1.1) and index (1.3), in (3.5) and (3.7) it is otherwise convenient to assess the closeness of $x(\cdot)$ and $w(\cdot)$. Here it is necessary to check that, for an appropriately selected functional υ of type (4.3), an assertion similar to Lemma 1 is satisfied.

Theorem 1 and the properties of the minimax solution of problem (2.3), (2.4) that were given in Section 2 enable us to conclude that functional equations (2.2) and (2.3) can (by analogy with the theory of ordinary differential games [1–4, 10, 11]) be treated as the main equations of differential games with hereditary information.

5. EXAMPLE

Suppose

$$n = 2,\quad x = (x_1, x_2) \in R^2,\quad h = \text{const} > 0,\quad t_0 = 0,\quad t_* = -h$$

We shall examine the differential game for a system with delay

$$dx_1[t]/dt = \beta x_2[t-h],\quad dx_2[t]/dt = \alpha x_1[t] + b(t)u + v,\quad 0 \le t \le T \tag{5.1}$$

where α and β are unknown numbers, and the function $b(t)$ is continuous and satisfies the conditions $|b(t)| \ge 1$ with $t \in [0, t']$ and $|b(t)| \le 1$ with $t \in [t', T]$ ($0 < \tau' \le T$), $|u| \le 1$ and $|\upsilon| \le 1$, with the quality index

$$\gamma_* = \max\{|x_2[t_1]|,\ |x_2[t_2]|\} + |x_1[T]| \tag{5.2}$$

where $t_1 \in [0, t')$ and $t_2 \in (t_1, t']$ are specified instants of time.

Let the matrix function

$$\Psi[\xi] = \begin{Vmatrix} \psi_{11}[\xi] & \psi_{12}[\xi] \\ \psi_{21}[\xi] & \psi_{22}[\xi] \end{Vmatrix}$$

be such that

$$\Psi[\xi] = 0 \text{ when } \xi < 0, \quad \Psi[0] = E, \quad \frac{d\Psi[\xi]}{d\xi} = \begin{Vmatrix} \beta\psi_{21}[\xi - h] & \beta\psi_{22}[\xi - h] \\ \alpha\psi_{11}[\xi] & \alpha\psi_{12}[\xi] \end{Vmatrix} \text{ when } \xi \geqslant 0$$

We put

$$y_j(t, x[t_*[\cdot]t]) = \begin{cases} \psi_{21}[t_j - t]x_1[t] + \psi_{22}[t_j - t]x_2[t] + \\ + \int\limits_t^{t+h} \psi_{21}[t_j - \tau]\beta x_2[\tau - h]d\tau, & \text{if } \ t < t_j; \quad j = 1,\ 2 \\ x_2[t_j], & \text{if } \ t_j \leqslant t \end{cases}$$

$$y_3(t, x[t_*[\cdot]t]) = \psi_{11}[T - t]x_1[t] + \psi_{12}[T - t]x_2[t] + \int\limits_t^{t+h} \psi_{11}[T - \tau]\beta x_2[\tau - h]d\tau$$

$$l = (l_1, l_2, l_3) \in R^3, \quad \mu(l) = \max\{|l_1| + |l_2|,\ |l_3|\}$$

$$\chi(t, x[t_*[\cdot]t],\ l) = l_1 y_1(t, x[t_*[\cdot]t]) + l_2 y_2(t, x[t_*[\cdot]t]) + l_3 y_3(t, x[t_*[\cdot]t]) +$$

$$+\begin{cases} -\int\limits_t^{t'} |l_1\psi_{22}[t_1 - \tau] + l_2\psi_{22}[t_2 - \tau] + l_3\psi_{12}[T - \tau]|(|b(\tau)| - 1)d\tau + \\ + \int\limits_{t'}^{T} |\psi_{12}[T - \tau](1 - |b(\tau)|)|d\tau, & \text{if } \ t < t' \\ \int\limits_t^{T} |\psi_{12}[T - \tau](1 - |b(\tau)|)|d\tau, & \text{if } \ t' \leqslant t \end{cases}$$

Then, the functional

$$\Gamma_*^0(t^0, x^0[t_*[\cdot]t^0]) = \max_{\mu(l) \leqslant 1} \chi(t^0, x^0[t_*[\cdot]t^0], l), \quad (t^0, x^0[t_*[\cdot]t^0]) \in G \tag{5.3}$$

will be the value of game (5.1), (5.2), which can be derived, for example, by construction from [7–9]. Functional (5.3) is the minimax solution of the following equation in co-invariant derivatives

$$\partial_t\varphi + \nabla_1\varphi\beta x_2[t - h] + \nabla_2\varphi\alpha x_1[t] + |\nabla_2\varphi|(1 - |b(t)|) = 0$$

where

$$\partial_t\varphi = \partial_t\varphi(t, x[t_*[\cdot]t]), \quad (\nabla_1\varphi, \nabla_2\varphi) = \nabla\varphi(t, x[t_*[\cdot]t])$$

provided that

$$\varphi(T, x[t_*[\cdot]T]) = \max\{|x_2[t_1]|,\ |x_2[t_2]|\} + |x_1[T]|$$

on the right-hand end. This can be verified, assuming, for example, that

$$P = Q = [-1,\ 1]$$

$$F^*(t, x[t_*[\cdot]t], q) = \{f = (\beta x_2[t - h],\ \alpha x_1[t] + b(t)u + q) \mid |u| \leqslant 1\}$$

$$F_*(t, x[t_*[\cdot]t], p) = \{f = (\beta x_2[t - h],\ \alpha x_1[t] + b(t)p + v) \mid |v| \leqslant 1\}$$

By the construction from Sections 3 and 4, and taking account of relations (5.1)–(5.3), we establish that the strategies

$$u_e^*(t, x[t_*[\cdot]t] = -\text{sign}[b(t)(l_1^u\psi_{22}[t_1 - t] + l_2^u\psi_{22}[t_2 - t] + l_3^u\psi_{12}[T - t])]$$

where

$$(l_1^u, l_2^u, l_3^u) \in \arg\max_{\mu(l)\leq 1}\{\chi(t, x[t_*[\cdot]t], l) - \varepsilon_u(t, x[t_*[\cdot]t])\mu(l)\}$$

$$\varepsilon_u(t, x[t_*[\cdot]t]) = \max\{0, \Gamma_*^0(t, x[t_*[\cdot]t]) - \Gamma_*^0(t^0, x^0[t_*[\cdot]t^0])\}$$

and

$$\upsilon_e^*(t, x[t_*[\cdot]t]) = \text{sign}[l_1^\upsilon\psi_{22}[t_1 - t] + l_2^\upsilon\psi_{22}[t_2 - t] + l_3^\upsilon\psi_{12}[T - t]]$$

where

$$(l_1^\upsilon, l_2^\upsilon, l_3^\upsilon) \in \arg\max_{\mu(l)\leq 1}\{\chi(t, x[t_*[\cdot]t], l) + \varepsilon_\upsilon(t, x[t_*[\cdot]t])\mu(l)\}$$

$$\varepsilon_\upsilon(t, x[t_*[\cdot]t]) = \max\{0, \Gamma_*^0(t^0, x^0[t_*[\cdot]t^0]) - \Gamma_*^0(t, x[t_*[\cdot]t])\}$$

will be optimal in game (5.1), (5.2).

Computer modelling of the control process with

$$\alpha = \beta = 1, \quad b(t) = 2 - t/2, \quad h = 1, \quad t_1 = \tfrac{1}{2}, \quad t_2 = \tfrac{3}{2}, \quad T = 4$$

$$t^0 = 0, \quad x^0[-1[\cdot]0] = \{(x_1^0[\tau] = -2, \ x_2^0[\tau] = 1 + \sin 2\pi\tau), \ -1 \leq \tau \leq 0\}$$

by uniform subdivision of the time interval [0, 4] with step $\delta = 0.01$ gave the following results.

The *a priori* estimated value of the game $\Gamma_*^0 = \Gamma_*^0(0, x^0[-1[\cdot]0]) \approx 0.939$. By the action of a pair of strategies $(u_e^*(\cdot), \upsilon_e^*(\cdot))$, the result $\gamma_* = 0.935 \approx \Gamma_*^0$ was obtained. By the action of $u_e^*(\cdot)$ in a pair with $\upsilon(\cdot) = \sin 4\pi t$, $\gamma_* = 0.638 < \Gamma_*^0$ was obtained. By the action of $\upsilon_e^*(\cdot)$ in a pair with $u(\cdot) = \sin 4\pi t$, $\gamma_* = 9.758 > \Gamma_*^0$ was obtained.

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